Discrete Monodromy, Pentagrams, and the Method of Condensation

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1 Introduction

The purpose of this paper is to point out some connections between:

1. The monodromy of periodic linear differential equations;

2. The pentagram map, which we studied in [**S1**] and [**S2**];

3. Dodgson's method of condensation for computing determinants;

We discovered most of these connections through computer experimentation.

1.1 Monodromy

Consider the second order O.D.E.

$$f''(t) + \frac{1}{2}q(t)f(t) = 0. \tag{1}$$

Here $q(t)$ is 1-periodic. If $\{f_1, f_2\}$ is a basis for the solution space of Equation 1 then there is some linear $T \in SL_2(\boldsymbol{R})$ such that $f_j(t+1) = T(f_j(t))$ for $j = 1, 2$. The trace $\mathrm{tr}(T)$, which is independent of basis, is sometimes called the *monodromy* of Equation 1. The ratio $f = f_1/f_2$ gives a smooth map from $\boldsymbol{R}$ into the projective line. Here q is given by the *Schwarzian derivative*:

$$q = \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2. \tag{2}$$

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Here is a discrete analogue of Equation 1. The *cross ratio* of 4 points $a, b, c, d \in \boldsymbol{R}$ is given by

$$x(a,b,c,d) = \frac{(a-c)(b-d)}{(a-b)(c-d)}. \tag{3}$$

A calculation shows that the quantity

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} x(f(t-3\epsilon), f(t-\epsilon), f(t+\epsilon), f(t+3\epsilon)) \tag{4}$$

converges to a multiple of q, when f is sufficiently smooth. Thus, the cross ratio is a discrete analogue of the Schwarzian derivative. Suppose we have an infinite n-periodic sequence $...q_n, q_1, q_2, ..., q_n, q_1, ...$ We can find points $..., f_1, f_2, f_3, ...$ in the projective line such that

$$x(f_j, f_{j+1}, f_{j+2}, f_{j+3}) = q_j \qquad \forall j \tag{5}$$

There will be a projective transformation T such that $f_{j+n} = T(f_j)$ for all j. The conjugacy class of T only depends on q. To obtain a numerical invariant, we can lift T to $SL_2(\boldsymbol{R})$ and take its trace. This quantity is a rational function in the variables $q_1, ..., q_n$.

A main focus of this paper is a discrete analogue for the third order case. This analogue involves infinite polygons in the projective plane. In analogy to the cross ratio we will define *projective invariants* of polygons in §3.1. We begin with an infinite sequence $..., x_1, x_2, ...$ of projective invariants having period $2n$. These invariants determine, up to a projective transformation, an infinite polygon which is invariant under a projective transformation. We call P a *twisted n-gon*. In other words, we have a map $P : \boldsymbol{Z} \to \boldsymbol{RP}^2$ and a projective transformation T such that $P(n+j) = T(P(n))$ for all j.

The monodromies Ω_1 and Ω_2 corresponding to T are rational functions of the variables $x_1, ..., x_{2n}$. Let $[\cdot]$ denote the floor function. In §2.1 we will define polynomials $O_1, ..., O_{[n/2]}, O_n$ and $E_1, ..., E_{[n/2]}, E_n$. We call these polynomials the *pentagram invariants*. We will express the monodromies explicitly in terms of the pentagram invariants:

$$\Omega_1 = \frac{(\sum_{k=0}^{[n/2]} O_k)^3}{O_n^2 E_n}; \qquad \Omega_2 = \frac{(\sum_{k=0}^{[n/2]} E_k)^3}{E_n^2 O_n}. \tag{6}$$

1.2 The Pentagram Map

Roughly, the *pentagram map* is the map which takes the polygon P to the polygon P', as indicated in Figre 1. In §4 we will give a precise definition, which expresses the pentagram map as a composition of two involutions α_1 and α_2.

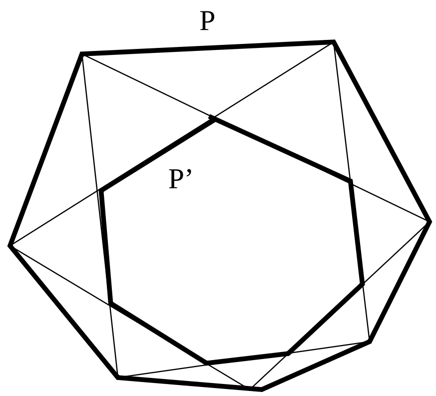

Figure 1

Expressed in our projective invariant coordinates—the cross ratio generalizations discussed in the previous section—the pentagram map has the form $\alpha_1(x_1, ..., x_{2n}) = (x'_1, ..., x'_{2n})$ and $\alpha_2(x_1, ..., x_{2n}) = (x''_1, ..., x''_{2n})$ where

$$x'_{2k-1} = x_{2k}\frac{1 - x_{2k+1}x_{2k+2}}{1 - x_{2k-3}x_{2k-2}}; \qquad x'_{2k} = x_{2k-1}\frac{1 - x_{2k-3}x_{2k-2}}{1 - x_{2k+1}x_{2k+2}};$$

$$x''_{2k+1} = x_{2k}\frac{1 - x_{2k-2}x_{2k-1}}{1 - x_{2k+2}x_{2k+3}} \qquad x''_{2k} = x_{2k+1}\frac{1 - x_{2k+2}x_{2k+3}}{1 - x_{2k-2}x_{2k-1}} \tag{7}$$

In these formulas, the indices are taken mod $2n$. We let $\alpha = \alpha_1 \circ \alpha_2$. In general, α has infinite order.

It turns out that the pentagram invariants are invariant polynomials for the *pentagram map*, when it is expressed in suitable coordinates.

Theorem 1.1 *$O_k \circ \alpha_j = E_k$ and $E_k \circ \alpha_j = O_k$ for $j = 1, 2$ and for all k.*

In §2 we will give a completely algebraic proof of Theorem 1.1. In §3-4 we will give a more conceptual proof which goes roughly as follows: The pentagram map commutes with projective transformations and therefore must preserve the monodromies Ω_1 and Ω_2. It follows from the general homogeneity properties of Equation 7 that the pentagram map must preserve the

properly weighted homogeneous pieces of the monodromies, and these pieces are precisely the pentagram invariants. In §6 we prove

Theorem 1.2 *The pentagram invariants are algebraically independent, so that α has at least $2[n/2]+2$ algebraically independent polynomial invariants.*

We conjecture that the pentagram invariants give the complete list of invariants for the pentagram map, at least when it acts on the spaces of twisted n-gons. We also conjecturethat the algebraic varieties cut out by the pentagram invariants are complex tori, after a suitable compactification. Finally we conjecture that the pentagram map acts on these complex tori as a translation in the natural flat metric.

1.3 The Method of Condensation

Let M be an $m \times m$ matrix. Let M_{NW} be the $(m-1) \times (m-1)$ minor obtained by crossing off the last row and column of M. Here N stands for "north" and W stands for "west". We define the other three $(m-1)\times(m-1)$ minors M_{SW}, M_{NE} and M_{SW} in the obvious way. Finally, we define M_C to be the "central" $(m-2) \times (m-2)$ minor obtained by crossing off all the extreme rows and columns of M. Dodgson's identity says

$$\det(M)\det(M_C) = \det(M_{NW})\det(M_{SE}) - \det(M_{SW})\det(M_{NE}). \tag{8}$$

Assuming that $\det(M_C)$ is non-zero, Equation 8 expresses $\det(M)$ as a rational function of determinants of matrices of smaller size. This procedure can be iterated, expressing the determinants of these smaller matrices as rational functions of determinants of still smaller matrices. And so on. This method of computing matrices is called *Dodgson's method of condensation*. See [**RR**] for a detailed discussion of this method and the rational functions that arise.

In §5 we will relate the pentagram map to the method of condensation. In some sense, *the pentagram map computes determinants*. We exploit this point of view to prove

Theorem 1.3 *Suppose that P is a $4n$-gon whose sides are alternately parallel to the x and y axes. Then (generically) the $(2n-2)$nd iterate of the pentagram map transforms P into a polygon whose odd vertices are all collinear and whose even vertices are all collinear.*

The surprise in Theorem 1.3 is that P could have trillions of sides. The pentagram map goes about its business for trillions of iterations and then the whole thing collapses all at once into a polygon whose vertices lie on a pair of lines. Theorem 1.3 is closely related to the main result in [**S3**], which we proved by geometric methods.

1.4 Paper Overview

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1.5 Acknowledgements

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2 The Invariants

2.1 Basic Definitions

All our definitions depend on a fixed integer $n \geq 3$. We will sometimes suppress n from our notation. Let $Z = \{1, 2, 3, ..., 2n\}$. We think of the elements of Z as being ordered cyclically, so that $2n$ and 1 are consecutive. Also, in our notation all our indices are taken cyclically.

We say that an *odd unit* of Z is a subset having one of the two forms:

1. $U = \{j\}$, where j is odd.
2. $U = \{k-1, k, k+1\}$, where k is even.

We say that two odd units U_1 and U_2 are *consecutive* if the set of odd numbers in the union $U_1 \cup U_2$ are consecutive. For instance $\{1\}$ and $\{3, 4, 5\}$ are consecutive whereas $\{1, 2, 3\}$ and $\{7, 8, 9\}$ are not.

We say that an *odd admissible subset* is a nonempty subset $S \subset X$ consisting of a finite union of odd units, no two of which are consecutive. We define the *weight* of S to be the number of odd units it contains. We denote this quantity by $|S|$. We define the *sign* of S to be the $+1$ is S contains an even number of singleton units, and -1 if S contains an odd number of singleton units. As an example, the subset

$$\{1, 5, 6, 7, 11\} = \{1\} \cup \{5, 6, 7\} \cup \{11\}$$

is an odd admissible subset if $n \geq 7$. This subset has weight 3 and sign $+1$. As an exception to this rule, we call the set $\{1, 3, 5, 7, ..., 2n-1\}$ odd admissible as well.

Each odd admissible subset S defines a monomial $O_S \in R$:

$$O_S = \text{sign}(S) \prod_{j \in S} x_j. \tag{9}$$

Let $O(k)$ denote the set of weight k odd admissible subsets of Z. If n is even then $O(k)$ is nonempty iff $k \in \{1, 2, ..., n/2, n\}$. If n is odd then $O(k)$ is nonempty iff $k \in \{1, 2, ..., (n-1)/2, n\}$. We define

$$O_k = \sum_{S \in O(k)} O_S. \tag{10}$$

By convention we set $O_0 = 1$.

We can make all the same definitions with the word *even* replacing the word *odd*. This leads to the definition of the E polymonials.

2.2 Proof of Theorem 1.2

Let $\alpha = \alpha_1 \circ \alpha_2$ be as in the introduction. For any rational function f, we define $\alpha(f) = f \circ \alpha$.

By definition

$$O_n = x_1 x_3 ... x_{2n-1}; \qquad E_n = x_2 x_4 ... x_{2n}. \tag{11}$$

If is easy to see directly from Equation 7 that $\alpha_j(O_n) = E_n$ and $\alpha_j(E_n) = O_n$. When n is even, we have

$$O_{n/2} = x_1 x_5 x_9 ... + x_3 x_7 x_{11} ...; \qquad E_{n/2} = x_2 x_6 x_{10} ... + x_4 x_8 x_{12} .... \tag{12}$$

Once again, it is easy to see directly from Equation 7 that $\alpha_j(O_{n/2}) = E_{n/2}$ and $\alpha_j(E_{n/2}) = O_{n/2}$. The interesting cases, which we now consider, are when $k < n/2$. We will show that $\alpha_1(O_k) = E_k$. The other cases have similar derivations.

Before we treat the general case we consider an example: We have

$$O_1 = \sum_{j=1}^{n} (-x_{2j+1} + x_{2j-1} x_{2j} x_{2j+1}).$$

Here indices are taken mod $2n$. We compute easily that

$$\alpha_1(x_{2j+1} - x_{2j-1} x_{2j} x_{2j+1}) = x_{2j+2} - x_{2k+2} x_{2j+3} x_{2j+4}. \tag{13}$$

Therefore

$$\alpha_1(O_1) = \sum_{j=1}^{n} (-x_{2j+2} + x_{2k+2} x_{2j+3} x_{2j+4}) = E_1.$$

This example suggests that the key to proving Theorem 1.2 lies in partitioning our polynomials in the right way.

Recall that $O(k)$ is the collection of weight k odd admissible sequences. Let $O_s(k) \subset O(k)$ consist of those sequences whose individual units are singletons. For instance $\{1, 5, 9\}$ is a member of $O_s(3)$ as long as $n \geq 6$. We call the odd units $\{j\}$ and $\{j-2, j-1, j\}$ *right partners*. We say that a sequence $S' \in O(k)$ is a *right partner* of a sequence $S \in O_s(k)$ if every odd unit in S has a right partner odd unit in S' and *vice versa*. For instance, the sequence $S = \{1, 7, 19\}$ and $S' = \{1, 5, 6, 7, 17, 18, 19\}$ are right partners. Also, S is a right partner with $\{1, 5, 6, 7, 19\}$. For any $S \in O_s(k)$ let $S_R \subset$

$O(k)$ be the subcollection of right partners. Every element of $O(k)$ has a unique right partner in $O_s(K)$. Therefore, we have a partition

$$O(k) = \bigcup_{S \in O_s(k)} S_R.$$

Correspondingly, we can write

$$O_k = \sum_{S \in O_s(k)} RO_S; \qquad RO_S = \sum_{S' \in S_R} O_{S'}.$$

We can make all the same definitions, with *even* replacing *odd* and *left* replacing *right*. Thus, we have a partition

$$E(k) = \bigcup_{S \in E_s(k)} S_L.$$

Here S_L consists of the set of left partners of S. Correspondingly we can write

$$E_k = \sum_{S \in E_s(k)} LE_S; \qquad LE_S = \sum_{S' \in S_L} E_{S'}.$$

Below we will prove

Lemma 2.1 *For any sequence $S \in S_s(k)$ there is a sequence $\overline{S} \in S_s(k)$ such that $\alpha_1(RO_S) = LE_{\overline{S}}$.*

Since α_1 is an involution the assignment $S \to \overline{S}$ is a bijection. Summing over the individual terms we have $\alpha(O_k) = E_k$. Thus, Lemma 2.1 implies Theorem 1.2.

To prove Lemma 2.1 we will decompose sequences in $S_s(k)$, which could be quite complicated, into much simpler sequences. We say that an odd admissible sequence is *tight* if it has the form

$$\{j, j+4, j+8, ..., j+4a\}.$$

As usual, these numbers are taken mod $2n$. Given any $S \in S_s(K)$ let T_S denote the set of maximal tight subsequences of S. For instance, if $S = \{3, 7, 19, 23, 35\}$ and $n = 18$ then T_S consists of the two sequences $\{19, 23\}$ and $\{35, 3, 7\}$. (The second sequence is congruent mod 36 to $\{35, 39, 43\}$.) Since $k < n/2$ every sequence in $S_s(k)$ decomposes in this way.

Lemma 2.2 *If $S \in S_s(k)$ then $RO_S = \prod_{T \in T_S} RO_T$.*

Proof: Let $T_S = \{T_1, ..., T_h\}$. Note that any number in T_i is at least 5 numbers away from any number in T_j. Otherwise, $T_i \cup T_j$ would be tight, contradicting maximality. From this observation we see that $T'_1 \cup ... \cup T'_h$ is odd-admissible for any choice of right partners $T'_1, T'_2, ..., T'_h$ of $T_1, ..., T_h$. Conversely, any right partner of S decomposes this way. Therefore S_R is precisely the union of the sets of the form $T'_1 \cup ... \cup T'_h$, where $T'_j \in (T_j)_R$ is arbitrary. Our lemma follows from this and from the distributive law. ♠

Lemma 2.3 *For any tight sequence T there is a tight sequence $\overline{T}$, having the same length, such that $\alpha_1(RO_T) = LE_{\overline{T}}$.*

Proof: Let $P = RO_T$ and let $P' = \alpha(P)$. Cyclically relabelling we can assume that $T = \{3, 7, 11, ..., 4a + 3\}$. The set T_R of right partners of T consists of T and the sequence $\{1, 2, 3, 7, 11, ..., 4a + 3\}$. Therefore

$$P = (1 - x_1x_2)x_3x_7x_{11}...x_{4a+3}.$$

Writing $x'_j = \alpha(x_j)$ we have

$$P' = (1 - x'_1x'_2)x'_3x'_7x'_{11}...x'_{4a+3}.$$

Using Equation 13 we see that $(1 - x'_1x'_2)x'_3 = x_4(1 - x_5x_6)$. We also have $x_5x_6 = x'_5x'_6$. Therefore

$$P' = x_4(1 - x'_5x'_6)x'_7, x'_{11}...x'_{4a+3}.$$

Using Equation 13 we see that $(1 - x'_5x'_6)x'_7 = x_8(1 - x_9x_{10})$. We also have $x_9x_{10} = x'_9x'_{10}$. Therefore

$$P' = x_4x_8(1 - x'_9x'_{10})x'_{11}, ..., x'_{4a+3}.$$

Continuing in this way we see that

$$P' = x_4x_8...x_{4a+4}(1 - x'_{4a+5}x'_{4a+6}) = x_4x_8...x_{4a+4}(1 - x_{4a+5}x_{4a+6}).$$

This last expression is exactly $LE_{\overline{T}}$, where $\overline{T} = \{4, 8, ..., 4a + 4\}$. ♠

Lemma 2.1 follows immediately from Lemma 2.2, Lemma 2.3, and the uniqueness of our decomposition. As we mentioned before, Theorem 1.2 follows from Lemma 2.1. This completes our proof.

3 Discrete Monodromy

3.1 PolyPoints and PolyLines

As in previous chapters we will fix some positive integer $n \geq 3$.

Let $\boldsymbol{P}$ be the projective plane over the field $\boldsymbol{F}$. Say that a *PolyPoint* is a bi-infinite sequence $A = \{...A_{-3}, A_1, A_5, ..\}$. of points in $\boldsymbol{P}$. (For technical reasons we always index these points by integers having the same odd congruence mod 4.) We assume also that there is a projective transformation T such that $T(A_j) = A_{j+4n}$ for all $j \in \boldsymbol{Z}$. We call T the *monodromy* of A.

Say that a *PolyLine* is a bi-infinite sequence $B = \{...B_{-1}, B_3, B_7, ..\}$ of lines in $\boldsymbol{P}$. We assume also that there is a projective transformation T such that $T(B_j) = B_{j+4n}$ for all $j \in \boldsymbol{Z}$. We call T the *monodromy* of B.

Given two points $a, a' \in \boldsymbol{P}$ we let (aa') be the line containing these two points. Given two lines $b, b' \in \boldsymbol{P}$ we let (bb') be the point of intersection of these two lines. Every PolyPoint A canonically determines a PolyLine B, by the rule $B_j = (A_{j-2}A_{j+2})$. At the same time every PolyLine B determines a PolyPoint A by the rule $A_j = (B_{j-2}B_{j+2})$. In this case we call A and B *associates*. By construction associates have the same monodromy.

The *dual space* to $\boldsymbol{P}$ is the space of lines in $\boldsymbol{P}$. This space, denoted by $\boldsymbol{P}^*$, is isomorphic to $\boldsymbol{P}$. Indeed $\boldsymbol{P}^*$ is the projectivization of the vector space dual to $\boldsymbol{F}^3$. Any projective transformation $T : \boldsymbol{P} \to \boldsymbol{P}$ automatically induces a projective transformation $T^* : \boldsymbol{P}^* \to \boldsymbol{P}^*$, and *vice versa*. Any point in $\boldsymbol{P}$ canonically determines a line in $\boldsymbol{P}^*$. Likewise, points in $\boldsymbol{P}^*$ canonically determine lines in $\boldsymbol{P}$ and lines in $\boldsymbol{P}^*$ canonically determine points in $\boldsymbol{P}$. The two spaces are on an equal footing.

Given the PolyPoint A, we define A^* to be the PolyPoint in $\boldsymbol{P}^*$ whose lines are given by the associate B. If the points of A are indexed by numbers congruent to 1 mod 4 then the points of A^* are indexed by numbers congruent to 3 mod 4, and *vice versa*. We make the same definitions for PolyLines. By construction $A^{**} = A$ and $B^{**} = B$. If T is the common monodromy of A and B then T^* is the common monodromy of A^* and B^*. We call A^* and B^* the *duals* of A and B.

For any projective transformation T, acting either on $\boldsymbol{P}$ or $\boldsymbol{P}^*$ we define

$$\Omega_1(T) = \frac{\mathrm{tr}^3(\widetilde{T})}{\det(\widetilde{T})}; \qquad \Omega_2(T) = \Omega_1(T^*). \tag{14}$$

Here $\widetilde{T}$ is a linear transformation whose projectivization is T. That is, $\widetilde{T}$

is a *lift* of T. It is easy to see that these quantities are independent of lift. Moreover, $\Omega_j(T)$ only depends on the conjugacy class of T. Finally, $\Omega_{3-j}(T^*) = \Omega_j(T)$ for any projective transformation.

If T is the monodromy of A we call $\Omega_1(T)$ and $\Omega_2(T)$ the *monodromy invariants* of A. By construction A^* has the same *set* of monodromy invariants as A, but their order is switched. The same goes for B. If S is some other projective transformation, then A and $S(A)$ have the same monodromy invariants. Likewise, B and $S(B)$ have the same monodromy invariants.

We now introduce our 2-dimensional versions of the cross ratio. If j is one of the indices for the points of A we define

$$p_{(j+1)/2}(A) = x(A_{j+8}, A_{j+4}, (B_{j+6}B_{j-2}), (B_{j+6}B_{j-6}))$$

$$q_{(j-1)/2}(A) = x(A_{j-8}, A_{j-4}, (B_{j-6}B_{j+2}), (B_{j-6}B_{j+6})) \tag{15}$$

Here x stands for the ordinary cross ratio, as in Equation 3. In the first equation, all 4 points lie on the line B_{j+6}. In the second equation, all 4 points lie on B_{j-6}. Conpare Figure 3 below. If the points of A are labelled by integers congruent to 1 mod 4 then the invariants of A are $...q_0, p_1, q_2, p_3, ...$ If the points of A are indexed by integers congruent to 3 mod 4 then the invariants of A are $...p_0, q_1, p_2, q_3, ...$ In this chapter we will only consider the case when the points of A are indexed by integers congruent to 1 mod 4, though in the next chapter we will consider both cases on an equal footing.

We can make all the same definitions for B, simply by interchanging the two roles of A and B in Equation 15. It turns out that our invariants are not just invariant under projective transformations, but also invariant under projective duality. Precisely, we have

$$p_j(A) = q_j(A^*); \quad q_j(A) = p_j(A^*); \quad p_j(B) = q_j(B^*); \quad q_j(B) = p_j(B^*) \tag{16}$$

for all relevant indices. To see this symmetry, we will consider an example.

Suppose that points of A are labelled by integers congruent to 1 mod 4. The first half of Figure 3 highlights the 4 points whose cross ratio is $p_3(A)$. The second half shows the lines whose cross ratio is used to define $q_3(A^*)$. The highlighted points are exactly the intersection points of the highlighted line with an auxilliary line. Hence, the two cross ratios are the same.

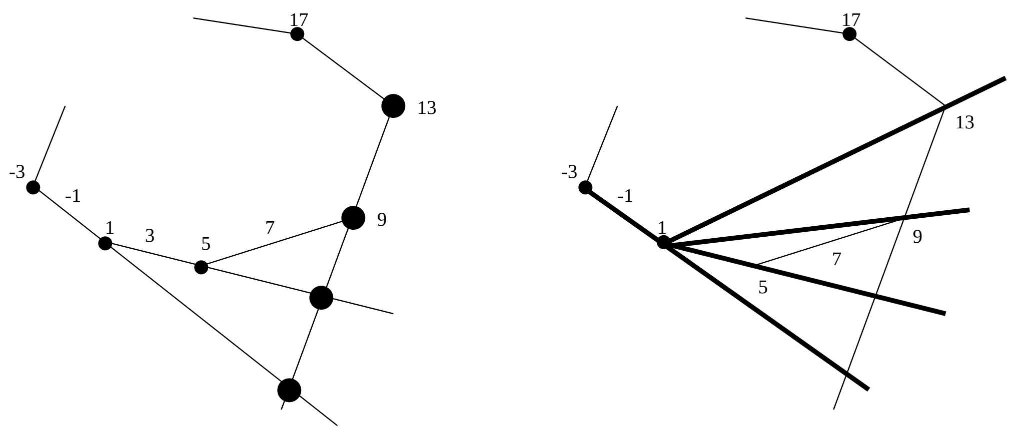

Figure 3

This chapter is devoted to establising Equation 6, which gives the formulas for Ω_1 and Ω_2 in terms of our invariants. Given the formula for Ω_1, the formula for Ω_2 follows from projective duality and from Equation 16. Thus, to establish Equation 6 it suffices to derive the equation for Ω_1.

3.2 Constructing the PolyPoint from its Invariants

In §2 we constructed our polynomials from the variables $x_1, ..., x_{2n}$. In this section we are going to use the alternate list of variables $p_1, q_2, p_3, q_4, ...$ The reason for the alternate notation is that it is useful to distinguish the even and odd variables in our constructions. The polynomials in §2 are obtained from the ones here using the substitution $p_i \to x_i$ when i is odd and $q_i \to x_i$ when i is even.

Suppose that $p_1, q_2, p_3, q_4, ...$ are given variables. We seek an infinite PolyPoint A such that

$$p_{2i-1}(A) = p_{2i-1}; \qquad q_{2i}(A) = q_{2i}; \qquad i = 1, 2, 3.... \tag{17}$$

What we mean by Equation 17 is that we wish to specify the points of A in such a way that the invariants we seek match a specified list $p_1, q_2, p_3, ...$ Likewise, we seek a formula for the associate B. For our purposes we only need the formulas for "half" of A and "half" of B. That is, we just need to know $A_{-3}, A_1, A_5, ...$ and $B_{-5}, B_{-1}, B_3, ...$

Here we make the same definitions as in §2.1, with respect $\boldsymbol{Z}$ (the integers) rather than the finite set Z. To each admissible sequence S we associate a monomial O_S in the formal power series ring $\boldsymbol{A} = \boldsymbol{Z}[[...p_1, q_2, p_3...]]$.

(Again, under the substitution mentioned above, the ring $\boldsymbol{A}$ is identified with $\boldsymbol{Z}[[...x_1, x_2, x_3, ...]]$.) For instance if $S = \{1, 2, 3, 9\}$ then $O_S = -p_1 q_2 p_3 q_9$. We count the empty subset as both even and odd admissible, and we define $O_\emptyset = E_\emptyset = 1$. Let O be the sum over all odd admissible sequences of finite weight. Likewise let E be the sum over all even admissible sequences of finite weight. We have $O, E \subset \boldsymbol{A}$. Given a pair of odd integers, (r, s) we define O_r^s to be the polynomial obtained from O by setting p_j equal to zero, for $j \leq r$ and $j \geq s$. We make the same definitions, with even replacing odd.

Let $A = \{...A_{-3}, A_1, A_5, ...\}$ and $B = \{...B_{-5}, B_{-1}, B_3, ...\}$, where (in homogeneous coordinates)

$$A_{-3} = [0,1,0]; \quad A_1 = [0,1,1]; \quad A_5 = [1,1,1];$$

$$A_{4j+1} = [O_1^{2j-1},\ O_{-1}^{2j-1} + p_1 O_3^{2j-1},\ O_{-1}^{2j-1}]; \quad j = 2,4,6... \tag{18}$$

$$B_{-5} = [0,0,1]; \quad B_{-1} = [1,0,0]; \quad B_3 = [0,1,-1]; \quad B_7 = [1,-1,0];$$

$$B_{4j+3} = [-E_2^{2j} + p_1 q_2 E_4^{2j},\ E_0^{2j},\ -E_0^{2j} + E_2^{2j}]; \quad j = 2,4,6... \tag{19}$$

In §5.2 we explicitly list out the first 7 points of A. We discovered these formulas as follows. We normalized the first few points of A and then found the equations for successive points using the definitions of the invariants. At some point we saw a pattern in the growing polynomials we were generating. The algebraic proofs we give in this section are really more like verifications. We did everything on the computer and simply converted our observations into a proof.

The basic tool for us is the following set of relations, which are easily derived.

$$\begin{array}{ll} O_r^s = 0 \quad \forall r > s; & O_{s-2}^s = O_s^s = 1; \\ E_r^s = 0 \quad \forall r > s; & E_{s-2}^s = E_s^s = 1; \end{array}$$

$$O_r^s = O_{r+2}^s - p_{r+2} O_{r+4}^s + P_{r+3} O_{r+6}^s; \quad r < s.$$

$$E_r^s = E_{r+2}^s - q_{r+2} E_{r+4}^s + Q_{r+3} E_{r+6}^s. \quad r < s.$$

$$O_r^s = O_r^{s-2} - p_{s-2} O_r^{s-4} + P_{s-3} O_r^{s-6}; \quad r < s.$$

$$E_r^s = E_r^{s-2} - q_{s-2} E_r^{s-4} + Q_{s-3} E_r^{s-6}; \quad r < s. \tag{20}$$

Here we have set

$$P_j = p_{j-1} q_j p_{j+1}; \qquad Q_j = q_{j-1} p_j q_{j+1}. \tag{21}$$

Let $\cdot$ stand for the dot product, and let $\times$ stand for the cross product.

Lemma 3.1 *Let $k \geq 2$ and $d \geq 0$. Then*

$$A_{4k+1} \cdot B_{4k+3+4d} = p_1 q_2 ... q_{2k} E_{2k+2}^{2k+2d}.$$

$$B_{4k+3} \cdot A_{4k+5+4d} = p_1 q_2 ... q_{2k} p_{2k+1} O_{2k+3}^{2k+1+2d}.$$

Proof: We will prove the first identity. The second one is very similar. We use the notation

$$(r, s) = O_r^{2k-1} E_s^{2k+2d}.$$

This notation should suggest to the reader that they plot the various *points* (r, s), given below, on a grid. The result is a neat graphical representation of the algebra we will be doing. Using Equations 18 and 19 we have

$$A_{4k+1} \cdot B_{4k+3+4d} = (-1, 2) - (1, 2) + p_1(3, 0) + p_1 q_2 (1, 4). \tag{22}$$

The basic relation

$$O_{-1}^* - O_1^* = -p_1 O_3^* + p_1 q_2 p_3 O_5^*,$$

implies that

$$(-1, 2) - (1, 2) = -p_1(3, 2) + p_1 q_2 p_3 (5, 2).$$

Plugging this into Equation 22 we have

$$A_{4k+1} \cdot B_{4k+3+d} = p_1((3, 0) - (3, 2) + q_2(1, 4) + q_2 p_3 (5, 2). \tag{23}$$

The basic relation

$$E_0^* - E_2^* = -q_2 E_4^* + q_2 p_3 q_4 E_6^*,$$

implies that
$$(3,0)-(3,2)=-q_2(3,4)+q_2p_3q_4(5,4).$$
Plugging this into Equation 23 gives
$$A_{4k+1}\cdot B_{4k+3+4d}=p_1q_2((1,4)-(3,4)+p_3(5,2)+p_3q_4(3,6)). \qquad (24)$$
note that Equation 24 has the same form as Equation 22 except that all the indices have been shifted by 2 and a factor of p_1q_2 appears. This process repeats until we reach:
$$A_{4k+1}\cdot B_{4k+3+4d}=p_1q_2...p_{2k-3}q_{2k-2}X,$$
where
$$X=(2k-3,2k+4)-(2k-1,2k+4)+p_{2k-1}(2k+3,2k)+p_{2k-1}q_{2k}(2k-1,2k+2)$$
$$=1-1+0+p_{2k-1}q_{2k}=p_{2k-1}q_{2k}.$$
The last two equations combine to give our identity. ♠

Lemma 3.2 *The following identities hold for all $k\geq 2$.*

1. $A_{4k+1}\times A_{4k+5}=p_1...p_{2k-1}B_{4k+3}$.
2. $B_{4k+3}\times B_{4k+7}=q_2...q_{2k}A_{4k+5}$.
3. $A_{4k+1}\cdot B_{4k+7}=p_1q_2...p_{2k-1}q_{2k}$.
4. $B_{4k+3}\cdot A_{4k+9}=p_1q_2...q_{2k}p_{2k+1}$.
5. $A_{4k+1}\cdot B_{4k+11}=p_1q_2...p_{2k-1}q_{2k}$.
6. $B_{4k+3}\cdot A_{4k+13}=p_1q_2...q_{2k}p_{2k+1}$.

Proof: We will derive these 6 identities from the two identities of Lemma 3.1. Taking $d=1$ we obtain Identities 3 and 4 listed above. Taking $d=2$ we obtain Identities 5 and 6 listed above. Taking $d=0$, we see that
$$A_{4k+1}\cdot B_{4k+3}=B_{4k+3}\cdot A_{4k+5}=0.$$

Thus, we may write

$$A_{4k+1} \times A_{4k+5} = \lambda_k B_{4k+3},$$

for some λ_k. An easy calculation verifies that $\lambda_2 = p_1 p_3$. Suppose, by induction, that $\lambda_{k-1} = p_1 ... p_{2k-3}$. We use the fact that

$$(A_{4k+5} \times A_{4k+1}) \cdot A_{4k-3} = (A_{4k+1} \times A_{4k+3}) \cdot A_{4k-5},$$

and the already proven identities show that $\lambda_k = p_1 ... p_{2k-1}$. The case for the B's is similar. ♠

Identity 1 says that A_i is the intersection point of the lines B_{i-2} and B_{i+2}. Identity 2 says that B_j is the line determined by A_{j-2} and A_{j+2}. Hence A and B are associates. Identities 3 and 4 say that generically $A_i \cdot B_{i-3} \neq 0$ and $B_j \cdot A_{j-3} \neq 0$. These statements also hold true in the few cases where some of the indices are negative, as may be verified by hand. Thus, A and B are in general position for generic choices of variables.

The first few identities of Equation $(*)$ can be verified by hand. We will show that $p_{2k+5}(P) = p_{2k+5}$ for $k \geq 2$. The case for the q's has a similar treatment. To aid us in the computation, we recall some vector identities:

$$A \times (B \times C) = -(A \cdot B)C + (A \cdot C)B$$

$$(A \times B) \times (A \times C) = ((A \times B) \cdot C)A. \tag{25}$$

If w_1, w_2, w_3, w_4 are vectors lying in the same 2-dimensional linear subspace of $\boldsymbol{F}^3$ we define

$$X(w_1, w_2, w_3, w_4) = \frac{(w_1 \times w_2) * (w_3 \times w_4)}{(w_1 \times w_3) * (w_2 \times w_4)}. \tag{26}$$

The operation $*$ means coordinate-wise multiplication. X will be a vector of the form (x, x, x). The number x is the classical cross ratio of the 4 points in the projective plane $\boldsymbol{P}$ represented by the vectors w_1, w_2, w_3, w_4.

By definition

$$p_{2k+5}(P) = X(A_{4k+17}, A_{4k+13}, B_{4k+15} \times B_{4k+7}, B_{4k+15} \times B_{4k+3}),$$

First:

$$A_{4k+17} \times A_{4k+13} = -p_1 ... p_{2k+5} B_{4k+15}.$$

Second:

$$(B_{4k+15} \times B_{4k+7}) \times (B_{4k+15} \times B_{4k+3}) = ((B_{4k+15} \times B_{4k+7}) \cdot B_{4k+3})B_{4k+15} =$$

$$((B_{4k+3} \times B_{4k+7}) \cdot B_{4k+15})B_{4k+15} = (q_2 ... q_{2k} A_{4k+5} \cdot B_{4k+15})B_{4k+15} =$$

$$(q_2 ... q_{2k})(p_1, q_2, ..., p_{2k+1} q_{2k+2})B_{4k+15}.$$

Third:

$$A_{4k+17} \times (B_{4k+15} \times B_{4k+7}) =$$

$$(A_{4k+17} \times B_{4k+7})B_{4k+15} - (A_{4k+17} \times B_{4k+15})B_{4k+7} =$$

$$-(p_1 q_2 ... p_{2k+2} q_{2k+3})B_{4k+15}.$$

Fourth:

$$A_{4k+13} \times (B_{4k+15} \times B_{4k+3}) =$$

$$-(A_{4k+13} \times B_{4k+15})B_{4k+3} + (A_{4k+13} \times B_{4k+7})B_{4k+15} =$$

$$(p_1 q_2 ... q_{2k} p_{2k+1})B_{4k+15}.$$

Notice that all the terms are multiples of the same vector. Using the formula for X, we have:

$$x = \frac{(p_1, p_{2k+5})(q_2 ... q_{2k})(p_1 q_2, ..p_{2k+1} q_{2k+2})}{q_{2k+2} p_{2k+3} (p_1 q_2, ..., p_{2k+1})^2} = p_{2k+5}.$$

3.3 The Final Calculation

Recall that T is the monodromy of A. We will first compute a lift of T to $GL_3(\boldsymbol{F})$. We may interpret an element in $GL_3(\boldsymbol{F})$ as a triple $\widetilde{T} = (V_1, V_2, V_3)$ of vectors in $\boldsymbol{F}^3$. The linear action of T is then given as follows: If $W = [w_1, w_2, w_3]$, then

$$\widetilde{T}(W) = w_1 V_1 + w_2 V_2 + w_3 V_3. \tag{27}$$

Consider the element $\widetilde{T} = (V_1, V_2, V_3)$, where

$$V_1 = p_1 A_{4n+5} - p_{2n+1} A_{4n+1};$$

$$V_2 = p_{2n-1} q_{2n} p_{2n+1} A_{4n-3};$$

$$V_3 = p_{2n+1} A_{4n+1} - p_{2n-1} q_{2n} p_{2n+1} A_{4n-3}. \tag{28}$$

Lemma 3.3 *$\widetilde{T}$ is a lift of T.*

Proof: Using Equations 27, 28 and 18 we compute

$$\widetilde{T}(A_{-3}) = \widetilde{T}[0,1,0] = p_{2n-1}q_{2n}p_{2n+1}A_{4n-3}.$$

$$\widetilde{T}(A_1) = \widetilde{T}[0,1,1] = p_{2n+1}A_{4n+1}.$$

$$\widetilde{T}(A_5) = \widetilde{T}[1,1,1] = p_1A_{4n+5}.$$

$$\widetilde{T}(A_9) = \widetilde{T}[1,1,1-p_1] =$$

$$p_1(A_{4n+5} - p_{2n+1}A_{4n+1} + p_{2n-1}q_{2n}p_{2n-1}A_{4n-3}) = p_1A_{4n+9}.$$

This last equality follows Equation 20, applied componentwise to our vectors. Our four computations show that the projectivization of $\widetilde{T}$ has the same action on the points A_{-3}, A_1, A_5, A_9 as T does. These 4 points are (for generic choice of variables) in general position, and projective transformations are determined by their action on 4 general position points. Hence the projective action of $\widetilde{T}$ coincides with the action of T. ♠

Now we compute Ω_1. We set

$$\widetilde{O} = \sum_{i=0}^{[n/2]} O_k.$$

Before we make the next calculation we note that $p_{2n+1} = p_1$, under the assumption that the invariants are $2n$-periodic. Now for the calculation:

$$\mathrm{tr}(T) = V_{11} + V_{22} + V_{33} =$$

$$(p_1O_1^{2n+1} - p_{2n+1}O_1^{2n-1}) +$$

$$(p_{2n-1}q_{2n}p_{2n+1}O_{-1}^{2n-3} + p_1p_{2n-1}q_{2n}p_{2n+1}O_3^{2n-3}) +$$

$$(p_{2n+1}O_{-1}^{2n-1} - p_{2n-1}q_{2n}p_{2n+1}O_{-1}^{2n-3}) =$$

$$p_1([O_1^{2n+1}] + [O_{-1}^{2n-1} - O_1^{2n-1}] + [p_{2n-1}q_{2n}p_1O_3^{2n-1}]) = p_1\widetilde{O}. \tag{29}$$

In the last line, we have bracketed terms so as to isolate the different kinds of terms in $\widetilde{O}$. The first expression describes the terms of $\widetilde{O}$ which

involve p_1, but not $p_{2n-1}q_{2n}p_1$. The second expression describes the terms of $\widetilde{O}$ which do not involve p_1. The third expression describes terms of $\widetilde{O}$ which involve $p_{2n-1}q_{2n}p_1$.

Again using the fact that $p_{2n+1} = p_1$ we have

$$\det(T) = (V_1 \times V_2) \cdot V_3 =$$

$$(p_1)(p_{2n-1})(p_{2n-1}q_{2n}p_{2n+1})((A_{4n-5} \times A_{4n-1}) \cdot A_{4n+3}) =$$

$$p_1^3(p_3 ... p_{2n+1})^2(q_2 ... q_{2n}) = p_1^3(p_1 p_3 ... p_{2n-1})^2(q_2 ... q_{2n}). \tag{30}$$

Combining Equations 29 and 30 we get the formula for Ω_1.

4 The Pentagram

4.1 Basic Definition

We fix n, as in previous chapters. Let P_1 (respectively P_2) be the space of PolyPoints which have $2n$ periodic invariant coordinates, and whose points are labelled by integers congruent to 1 (respectively 3) mod 4. Likewise we define L_1 and L_2 for PolyLines.

Define

$$X = P_1 \cup P_2 \cup L_1 \cup L_2. \tag{31}$$

Suppose that $A = \{A_j\}$ is a PolyPoint. We define $\delta_1(A) = \{B_j\}$ where

$$B_j = (A_{j-2}A_{j+2}). \tag{32}$$

$\delta_1(A)$ is just the associate of A. We make the same definition for PolyLines. δ_1 is an involution of X which interchanges spaces P_1 and L_3 and interchanges P_3 and L_1.

We define $\delta_1(A) = \{B'_j\}$ where

$$B_j = (A_{j-4}A_{j+4}). \tag{33}$$

δ_2 is an involution of X which interchanges spaces P_1 and L_1 and interchanges P_3 and L_3. Figure 5.1 shows the action of δ_2 on a PolyPoint in P_1.

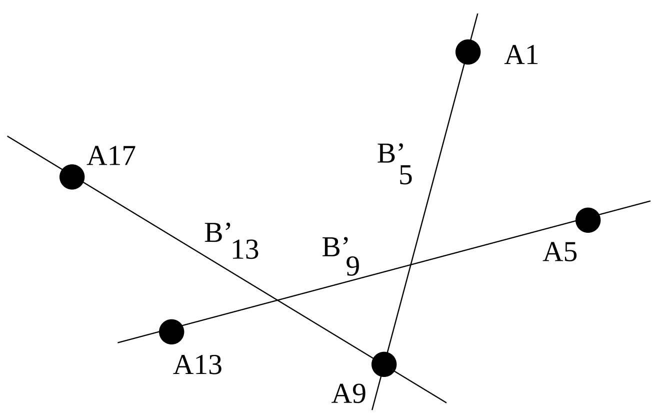

Figure 5.1

We define

$$\alpha_1 = \delta_1 \circ \delta_2 \circ \delta_1; \qquad \alpha_2 = \delta_2. \tag{34}$$

We call the pair (α_1, α_2) the *pentagram map*. Both α_1 and α_2 are involutions. Moreover, conjugation by δ_1 interchanges these two maps.

4.2 The Pentagram Map in Coordinates

There is a "forgetful map" which takes the variables p_j and q_j and calls them x_j, regardless of the letter. There is also a map which takes a polygon and sends it to its invariant coordinates. Composing this coordinate map with the forgetful map we get generically defined bijections from any of our 4 spaces to $\boldsymbol{F}^{2n}$. In this way we can work out the pentagram map in the same coordinates used in Equation 7.

We can exploit symmetry to reduce the amount of computing we have to do. First, it is a consequence of Equation 16 and the invariants of our coordinates under projective duality that

1. The action of α_j on P_j is the same as the action of α_j on L_j, and this action does not depend on j.
2. The action of α_j on P_{3-j} is the same as the action of α_j on L_{3-j}, and this action does not depend on j.

We will show that Equation 7 describes the action of α_1 and α_2 on $P_1 \cup L_1$. If we computed the action on $P_2 \cup L_2$ we would have to interchange α_1 with α_2 to get the right formulas. We will compute the action of α_2 on P_1. The derivation for α_1 is similar and follows from symmetry. To see that the formula in Equation 7 is correct, we just have to work out a single invariant of $\alpha_2(P)$. The formulas for the other invariants are forced by the dihedral symmetry of our constructions.

We will not do this calculation by hand. However, we will set it up exactly, so that the inclined reader can type everything into a symbolic manipulator and push the button, as we did. Let A be as in §4. We will compute the invariant $q_4' = q_4(B')$, which is the variable x in the expression $X = (x, x, x)$, where (as in Equation 26)

$$X = X(B_1', B_5', A_3' \times A_{11}', A_3' \times A_{15}'). \tag{35}$$

Here A' is the associate of B', so that

$$A_3' = B_1' \times B_5'; \quad A_{11}' = B_9' \times B_{13}'; \quad A_{15}' = B_{13}' \times B_{17}'. \tag{36}$$

Finally, we have

$$B_1' = A_{-3} \times A_5; \quad B_5' = A_1 \times A_9; \quad ... \quad B_{17}' = A_{13} \times A_{21}. \tag{37}$$

To make our computation we need to know the 7 points $A_{-3}, ..., A_{21}$. Once we have these 7 points, we just take a bunch of cross products.

Using the convention of Equation 21 we list these points explicitly.

$$A_{-3} = [0,1,0]; \quad A_1 = [0,1,1]; \quad A_5 = [1,1,1]; \quad A_9 = [1,1,1-p_1];$$

$$A_{13} = [1-p_3, 1-p_3+Q_2, 1-p_1-p_3+Q_2];$$
$$A_{17} = [1-p_3-p_5+Q_4, 1-p_3-p_5+Q_2+Q_4, 1-p_1-p_3-p_5+p_1p_5+Q_2+Q_4];$$

$$\begin{gathered} A_{21} = [1-p_3-p_5-p_7+p_3p_7+Q_4+Q_6, \\ 1-p_3-p_7-p_9+Q_2+Q_4+Q_6+p_3p_7-p_7Q_2, \\ 1-p_1-p_3-p_7+Q_2+Q_4+Q_6+p_1p_5+p_1p_7+p_3p_7-p_1Q_6-p_7Q_2] \end{gathered}$$

We double checked these formulas using Equation 20.

When we plug everything into Mathematica and compute, we get

$$q_4' = p_5 \frac{1-q_6p_7}{1-q_2p_3}.$$

Forgetting about the lettering, we have

$$x_4' = x_5 \frac{1-x_6x_7}{1-x_2x_3}.$$

One can see that this exactly matches the equation for α_2 given in Equation 7, for $j=2$. The general case follows from dihedral symmetry.

4.3 Second Proof of Theorem 1.2

Suppose A is a PolyPoint, with invariants $p_1, q_2, ..., q_{2n}$, as above. Let T be the projective transformation such that $T(A_{j+2n}) = A_j$. Let Ω_1 and Ω_2 be the two monodromy invariants of A.

We will just consider α_1. The proof for α_2 is the same. Let $B' = \alpha_1(A)$. Our constructions commute with projective transformations. Hence B' is also invariant under T. This is to say that the dual of B', which is another PolyPoint, is invariant under T^*, the dual of T. Hence $\alpha_1(\Omega_j) = \Omega_{3-j}$. It now follows from Equation 6 that

$$\alpha_1\Big(\frac{(\sum_{k=0}^{[n/2]} O_k)^3}{O_n^2 E_n}\Big) = \frac{(\sum_{k=0}^{[n/2]} E_k)^3}{E_n^2 O_n}. \tag{38}$$

One can see easily from Equation 7 that $\alpha_1(O_n) = E_n$ and $\alpha_1(E_n) = O_n$. Therefore

$$\alpha_1\Big(\sum_{k=0}^{[n/2]} O_k\Big) = \sum_{k=0}^{[n/2]} E_k. \tag{39}$$

Now for the moment of truth. Let $S_t : \boldsymbol{F}^{2n} \to \boldsymbol{F}^{2n}$ be as in Equation 57. Looking at Equation 7 we see that

$$\alpha_1 \circ S_t = S_{t^{-1}} \circ \alpha_1. \tag{40}$$

At the same time, we have

$$O_k \circ S_t = t^{-k} O_k; \qquad E_k \circ S_t = t^k E_k. \tag{41}$$

Therefore

$$\alpha_1\Big(\sum_{k=0}^{[n/2]} t^k O_k\Big) = \sum_{k=0}^{[n/2]} t^k E_k. \tag{42}$$

Since this last equation is true for all t, we must have $\alpha_1(O_k) = E_k$ for all k. Since α_1 is an involution, $\alpha_1(E_k) = O_k$ for all k. This completes our proof.

4.4 Conic Sections

In this section we establish a technical result used in later sections. Namely, if A is inscribed in a conic then $E_n(A) = O_n(A)$. We continue using the notation established above, and take $A \in P_1$.

Lemma 4.1 *Suppose A is inscribed in a conic. Then*

$$(1 - q_{2k})(1 - q_{2k-2}p_{2k-1}) = (1 - p_{2k-1})(1 - q_{2k}p_{2k+1});$$

$$(1 - p_{2k-1})q_{2k}(1 - p_{2k+1}) = (1 - q_{2k-2})p_{2k-1}(1 - q_{2k})$$

for $k = 1, ..., n$. Here indices are taken mod $2n$.

Proof: We will derive the first identity. The second one is a fairly straightforward rearrangement of the first. Figure 5.2 shows an application of Pascal's theorem. If the 6 points $A_{-3}, A_1, A_5, A_9, A_{13}, A_{17}$ lie on a conic section then the points C_1, C_2, C_3 lie on a line.

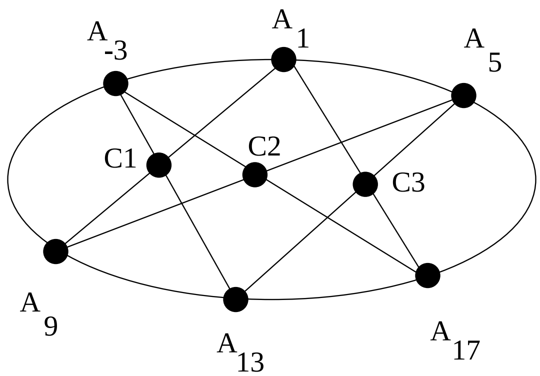

Figure 5.2

If we express the C_j in homogeneous coordinates, we have

$$C_1 = (A_{-3} \times A_{13}) \times (A_1 \times A_9)$$

$$C_2 = (A_{-3} \times A_{17}) \times (A_5 \times A_9)$$

$$C_3 = (A_5 \times A_{13}) \times (A_1 \times A_{17}) \tag{43}$$

The condition that the Cs lie on the same line is given by

$$\det(C_1, C_2, C_3) = 0. \tag{44}$$

In other words, we arrange these vectors into a 3×3 matrix and set the determinant equal to 0.

When we do this calculation on Mathematica, we see that Equation 44 holds if and only if $(1 - q_4)(1 - q_2 p_3) = (1 - p_3)(1 - q_4 p_5)$. If all points of A lie on the same conic, then by symmetry the same relation holds with the indices shifted by $2, 4, 6$... This completes our proof. ♠

Corollary 4.2 *If A is inscribed in a conic then $O_n(A) = E_n(A)$.*

Proof: Taking the product of the first equation over all k, and cancelling terms, we see that $\prod(1 - p_{2k-1}) = \prod(1 - q_{2j})$. Taking the product of the second equation, over all k, we get $O_n(\prod(1 - q_{2k})^2 = E_n(\prod(1 - p_{2k-1})^2$. Using the first equation to cancel terms, we are left with $O_n = E_n$. ♠

5 The Method of Condensation

5.1 Octahedral Tilings

There is a tiling T of $\boldsymbol{R}^3$ by octahedra, which can be described as follows. The vertices of T have the form $(a,b,c) \in \boldsymbol{Z}^3$, subject to the constraints that the three coordinates are either all even or all odd. Two vertices are joined by an edge if their distance is exactly $\sqrt{3}$. One of the octahedra in T has the 6 vertices $(0,0,0)$, $(\pm 1, \pm 1, 1)$ and $(2,0,0)$. All the other octahedra are translates of this one. In our model octahedron, we call $(2,0,0)$ the *top* and call $(0,0,0)$ the *bottom*. We call $(-1,1,1)$ the *northwest vertex*, $(1,1,1)$ the *northeast vertex* and so forth. We extend this definition to all octahedra using translations.

Suppose that M is an $m \times m$ matrix. Dodgson's method of condensation involves the connected square minors of M. Suppose that

$$H = \{M_{ij} |\ a_1 \leq i \leq a_2;\ b_1 \leq j \leq b_2\} \tag{45}$$

is such a minor. We define $f(H) = (a,b,c)$ where

$$a = a_1 + a_2; \qquad b = b_1 + b_2; \qquad c = a_1 - a_2 = b_1 - b_2. \tag{46}$$

For instance, if H is the singleton M_{11} then $f(H) = (2,2,0)$. If $H = M$ then $f(H) = (m+1, m+1, m-1)$. We let $[M]$ denote the set of vertices of the tiling which have the form $f(H)$. Note that $[M]$ looks like a pyramid with square base.

We can label the point $f(H) \in [M]$ by $\det(H)$. Dodgson's identity gives a single relation for each octahedron:

$$V_t V_b = V_{nw} V_{se} - V_{sw} V_{ne}. \tag{47}$$

Here V_t is the label of the top vertex, V_b is the label of the bottom vertex, and so forth. To compute $\det(M)$, begins at the base of the pyramid and computes successive layers using the octahedron rule. At the end, one arrives at the apex of the pyramid, with the final answer.

What we do next works most gracefully when $\det(M) \neq 0$. We say that a labelling of a horizontal layer of T is *constant* if every vertex gets the same label. We say that a *sandwich condensation* is a labelling of all the layers of T between two constant horizontal layers. Here is an example: We label the layer $\{z=0\}$ by 1's. Next, we label the layer $\{z=1\}$ so as to be doubly

periodic, with period m in each direction. In one $m \times m$ block, we put the entries of M. The second layer will be labelled by determinants of 2×2 connected minors of M and its cyclic permutations. By Dodgson's identity, the third layer will be labelled by the determinants of the 3×3 connected minors of M and its cyclic permutations. And so forth. The mth layer will be labelled by the determinants of M and its cyclic permutations. But all these determinants have the same value. This is our sandwich. It has *width* m.

To consider a more generic situation, we say that a *condensation* is a labelling of the vertices of the tiling T, such that the labelling satisfies the octahedron rule at every octahedron. The labellings on two successive horizontal layers determine the condensation. If these labels are chosen generically, in $\boldsymbol{C}$ for example, then we will not encounter singularities when trying to propagate the condensation to the other layers, both above and below the two initial ones.

5.2 Picture of the Pentagram Map

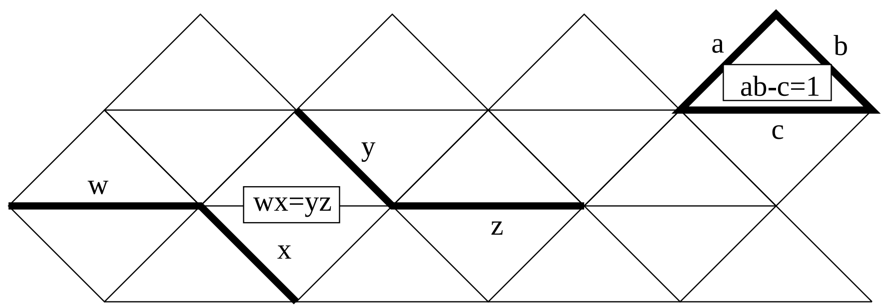

Figure 5.1

Figure 5.1 shows a tiling of the plane by isosceles triangles. Suppose one labels the edges of the tiling with elements of a field, in a way which is invariant under a horizontal translation, subject to the compatibility rules indicated in the figure. (These compatibility rules are supposed to hold for all configurations isometric to the ones highlighted.) We call such a labelling a *pentagram labelling*.

Given a pentagram labelling, the maps α_1 and α_2, the two involutions from Equation 7, express how one one deduces the labellings on a given row from the labellings on the rows above or below it. Thus, an orbit of the pentagram map is encoded by a pentagram labelling.

5.3 Circulent Condensations

Consider the linear projection $\pi : \boldsymbol{R}^3 \to \boldsymbol{R}^2$ given by

$$\pi(x, y, z) = (2x - y, z). \tag{48}$$

We say that a condensation is *circulent* to π if every vertex in a fiber $\pi^{-1}(p)$ gets the same label. Referring again to Dodgson's method of condensation, the circulent labellings correspond to certain circulent matrices.

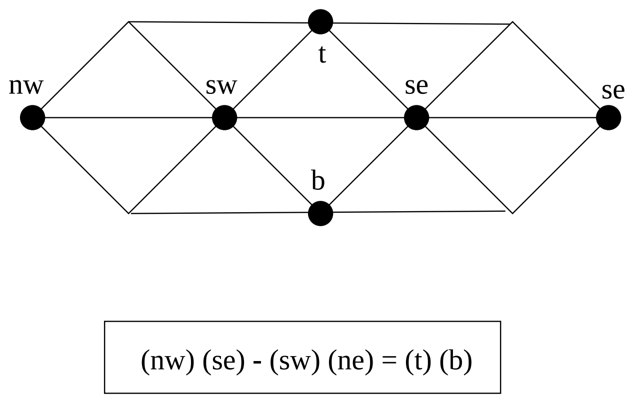

Figure 5.2

We can use π to push a circulent condensation into the plane, without losing any information. π maps the vertices of T to the vertices of the tiling τ shown in Figure 5.1. Figure 5.2 shows the image of a single octahedron under π, superimposed onto τ. Figure 5.2 also shows the local rule satisfied by the image of an circulent condensation.

We can think of an circulent condensation as a labelling of the vertices of τ which satisfies the local condition shown in Figure 5.2. Figure 5.3 shows how to convert from an circulent condensation to a pentagram labelling.

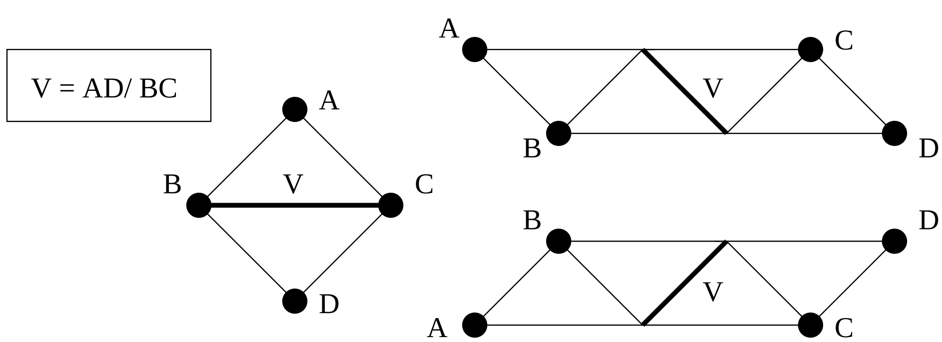

Figure 5.3

To verify the first of the two compatibility rules for pentagram labellings we refer to Figure 5.4 and compute

$$ab - c = \frac{CD}{AF}\frac{AG}{CE} - \frac{BH}{EF} = \frac{DG - BH}{EF} = 1.$$

The last equality is the compatibility rule for the circulent condensation.

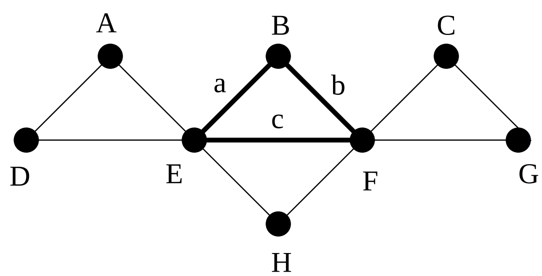

Figure 5.4

To verify the second compatibility rule we refer to Figure 5.5 and compute

$$wx = \frac{AI}{EF}\frac{EK}{GI} = \frac{AK}{FG} = \frac{AH}{CF}\frac{CK}{GH} = yz.$$

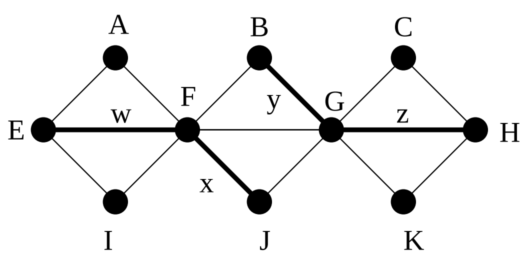

Figure 5.5

5.4 The Lifting Problem

As usual, suppose that n is fixed, as well as a suitable base field. Let C denote the space of circulent condensations which are periodic under a horizontal translation which shifts the labels by n. Let P denote the space of pentagram labels which are periodic under the same translation. The translation given in the previous section gives a map $\psi : C \to P$. In this section we ask about the extent to which ψ is invertible.

An element of C is determined by the values it attains on two successive horizontal rows. An element of P is determined by the values it takes on the non-horizontal edges of a single row. In both cases one puts down $2n$

values before the period repeats. That is, both C and P are $2n$ dimensional. The map ψ is a quotient map and certainly not 1-to-1. For instance, all the scalar multiples of a single element of C get mapped to the same element of P. Since C and P have the same dimension and ψ is far from injective, ψ is also far from surjective.

To give a simple and clean answer we will assume that $n = 4m$ for some $m \in \boldsymbol{N}$. The other cases have a somewhat messier analysis. Given an element $X = (x_1, ..., x_{2n}) \in P$ we have the pentagram invariants defined in §2.1.

Theorem 5.1 *If $n = 4m$ than an element $X \in P$ is contained in the image of ψ if and only if $O_{2m}(X) = E_{2m}(X) = 2$ and $O_n(X) = E_n(X) = 1$.*

Proof: For $j = 1, 2, 3, 4$ we define

$$f_j = \prod_{i \equiv j \bmod 4} x_i. \tag{49}$$

It is not hard to see that $O_{2m} = f_1 + f_3$. Also $O_n = f_1 f_3$. By hypotheses, we have $f_1 + f_3 = 2$ and $f_1 f_3 = 1$. This forces $f_1 = f_3 = 1$. Likewise $f_2 = f_4 = 1$. Conversely, if $f_j = 1$ for $j = 1, 2, 3, 4$ then we have the hypothesis of this theorem. All in all, the hypothesis of this theorem are equivalent to the hypothesis that $f_j = 1$ for $j = 1, 2, 3, 4$. We will work with this latter hypothesis.

First we prove necessity. Suppose X is in the image of ψ. We focus on a single row of our triangulation. Let c_k be the label of the vertex which is incident to the edges labelled $x_{k-1/2}$ and $x_{k+1/2}$. Here k is a half-integer. We have

$$x_1 = \frac{c_{-3/2} c_{7/2}}{c_{-1/2} c_{5/2}}; \quad x_5 = \frac{c_{5/2} c_{15/2}}{c_{7/2} c_{13/2}}; \quad x_9 = \frac{c_{13/2} c_{23/2}}{c_{15/2} c_{21/2}} \quad ... \tag{50}$$

When we compute f_1, each c-term appears exactly once in the numerator and exactly once in the denominator. Hence $f_1 = 1$. A similar argument shows that $f_2 = f_3 = f_4 = 1$.

To prove sufficiency, suppose that $f_1 = f_2 = f_3 = f_4 = 1$. We claim that there exist labels $r_1, r_2, ..., r_{2n}$ such that

$$x_j = \frac{r_{j+2}}{r_{j-2}}. \tag{51}$$

To see this, we pick r_1 arbitrarily. Then set, successively

$$r_5 = x_3 r_1; \quad r_9 = x_7 x_3 r_1; \quad r_{11} = x_{11} x_7 x_3 r_1 \quad ... \tag{52}$$

Since $n \equiv 0$ mod 4 when we cycle through one period we return back to the value $r_1 f_3 = r_1$. Thus, we can make a completely consistent choice of r for all indices congruent to 1 mod 4. The same argument works for the other congruences.

Given the r's, it suffices to find c's such that

$$r_j = \frac{c_{j-1/2}}{c_{j+1/2}}. \tag{53}$$

In our construction of the r's we had a free choice for each congruence mod 4. Thus, (in the generic case) we can choose the r's so that

$$\rho_j = \prod_{i \equiv j \mathrm{mod} 4} r_i = 1 \tag{54}$$

for $j = 1, 2, 3, 4$. Finding the c's in terms of the r's is exactly the same problem as finding the r's in terms of the x's. The ρ's play the same role as the f's. Thus, we can find our c's, which gives a preimage of X in C. ♠

5.5 Degenerate polygons

We fix $n = 4m$. To avoid trivial cases, we assume that m is large−say, $m \geq 3$. We work over $\boldsymbol{C}$. Everything we do is understood to be defined for the generic example, but perhaps not for every example. The reader should insert this caveat before every assertion. We will work with PolyPoints and PolyPoints which are periodic mod n. By this we mean that the points or lines themselves are periodic, not just the projective invariants. We will call such objects *periodic PolyPoints* or *periodic PolyLines*. We take these to be elements of the spaces P_1 and L_1 introduced in §5.1.

We say that the periodic PolyPoint $A = \{...A_1, A_5, ...\}$ is *degenerate* if $...A_1, A_9, A_{13}...$ lie on the same line and if $...A_3, A_{11}, A_{15}, ...$ all lie on the same line. We make the dual definition for PolyLines. A polygon (in the ordinary sense) satisfying the hypothesis of Theorem 1.3 naturally determines a degenerate PolyLine. One simply considers the lines considered by the edges of the polygon.

Lemma 5.2 *A periodic element in P_1 is degenerate iff its projective invariants satisfy $q_j p_{j+1} = 1$ for all j. Dually, a periodic element in L_1 is degenerate if and only if its projective invariants satisfy $p_j q_{j+1} = 1$ for all j.*

Proof: By duality it suffices to prove this result for PolyPoints. Referring to the PolyPoint constructed in §4, we compute

$$\det(A_{-1}, A_5, A_{13}) = p_1(1 - q_2 p_3).$$

The three points in question lie on the same line iff $q_2 p_3 = 1$. By symmetry, similar statements hold for other triples of points. ♠

Lemma 5.3 *If A is a periodic $2n$-gon then*

$$(\sum_{k=0}^{[n/2]} O_k)^3 = 27 O_n^2 E_n; \qquad (\sum_{k=0}^{[n/2]} E_k)^3 = 27 E_n^2 O_n.$$

Proof: This is a corollary of Equation 6. If A is $2n$-periodic then its monodromy T is the identity. Likewise, T^* is the identity matrix. Hence $\Omega_1 = \Omega_2 = 27$. ♠

For Theorem 1.3 we only need the second statement of the next result. The spaces P_1 and L_1 are defined in §5.1.

Lemma 5.4 *A degenerate element in either P_1 or L_1 has pentagram invariants*

1. *$O_k = E_k = 0$ for $k < 2m$ and*
2. *$O_{2m} = E_{2m} = 2$ and $O_n = E_n = 1$.*

Proof: First suppose that $k < 2m = n/2$. Using the reasoning in the proof of Theorem 1.1 we can group O_k into pieces having the form $\prod \mu$, where each μ is a monomial times $(1 - q_j p_{j+1})$ for some j. (Compare Lemma 2.2, but interchange p with q.) By Lemma 5.2 each μ is zero and hence $O_k = 0$. The same argument works for E_k.

For the second item note that $O_n E_n = (p_2 q_3)(p_4 q_5)... = (1)(1)... = 1$. Elements in P_1 or L_1 are degenerate limits of polygons which are either inscribed

in, or superscribed about, a conic section. Thus, Lemma 4.2 applies to these elements, by continuity. It now follows from Lemma 4.2 that $E_n^2 = O_n^2 = 1$. If $O_n = -1$, for any particular example, then by continuity O_n is identically -1. However, we can choose a $4n$-point, contained in $\boldsymbol{R}^2$, which is essentially the double cover of a $2n$-point. The invariants of such a $4n$-point would repeat with period $2n$, forcing O_n to be a square of a real number. This rules out the possibility that $O_n = -1$. Hence $O_n = 1$ for every example. Likewise $E_n = 1$ for every example. If now follows from Corollary 5.3, and from the vanishing of all the other pentagram invariants, that $1 + O_{2m} = 3O_n = 3$. Hence $O_{2m} = 2$. Likewise $E_{2m} = 2$. ♠

5.6 Proof of Theorem 1.3

We start with a degenerate PolyLine. This PolyLine determines a labelling of a single row of the triangulation τ, considered in §6. Combining Theorem 5.1 and Lemma 5.4 we see that we can find a circulent condensation C which translates into our pentagram labelling.

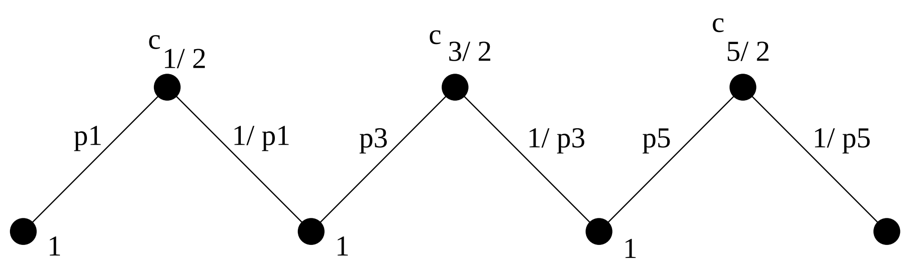

Figure 5.5

Referring to the proof of Theorem 5.1 we can use the fact that $p_j q_{j+1} = 1$ to arrange so that $r_j r_{j+1} = 1$ for all j. This allows to pick our c labelling so that the row below the edge labels is identically 1, as shown in Figure 5.5. Figure 5.5 is supposed to be periodic in the horizontal direction.

We pull our c labelling back to give a labelling of T, the tiling of $\boldsymbol{R}^3$ by octahedra. The bottom row of dots pulls back to an infinite horizontal plane of dots labelled by 1. The next layer up pulls back to give the labelling of the next horizontal plane, as shown in Figure 5.6. This labelling is periodic with respect to horizontal and vertical translation by n. Each $n \times n$ block is a circulent matrix. For generic choice of variables, this circulent matrix will have nonzero determinant.

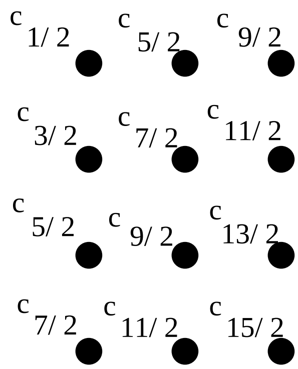

Figure 5.6

When we develop the condensation upwards, we arrive precisely at a sandwich condensation of width n. Going back to the planar picture, we see that the nth horizontal row of our vertex labelling is a constant labelling, as shown in Figure 5.7.

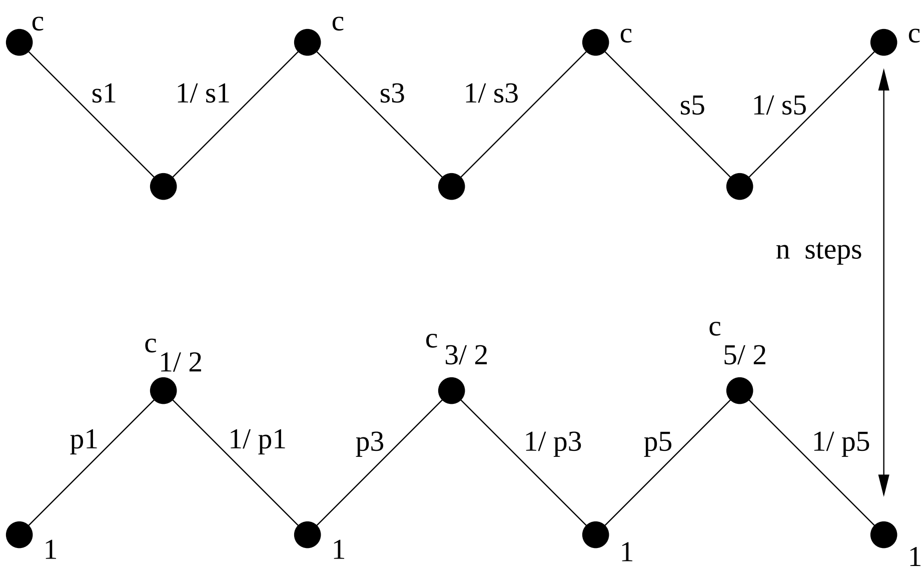

Figure 5.7

Since the top later of vertices is all constant, the layer of edges directly below it must have labels $s_1, t_2, s_3, t_4 ...$ where $t_2 = 1/s_1$, $t_4 = 1/s_3$, etc. By Lemma 5.2 the PolyPoint corresponding to this row is degenerate. That is, the vertices of this PolyPoint lie on a pair of lines. Translating this information back into the language of the pentagram map, we see that it corresponds exactly to the statement of Theorem 1.3.

6 Proof of Theorem 1.2

6.1 Proof modulo the Vanishing Lemma

In this chapter we prove that the pentagram invariants, constructed in §2.1, are algebraically independent. We take n odd, so that there are $n+1$ dihedral invariants. The even case is very similar.

Given a polynomial map $f : \boldsymbol{C}^{2n} \to \boldsymbol{C}$, let $\nabla f = (\partial f/\partial z_1, ..., \partial f/\partial z_{2n})$.

Lemma 6.1 *Suppose there is a $p \in \boldsymbol{C}^{2n}$ such that $\nabla f_1(p), ..., \nabla f_{n+1}(p)$ are linearly independent over $\boldsymbol{C}$. Then $f_1, ..., f_{n+1}$ are algebraically independent.*

Proof: For the sake of contradiction assume that there is some nontrivial $F \in \boldsymbol{Z}[x_1, ..., x_{n+1}]$ such that $F(f_1, ..., f_{n+1}) = 0$. By continuity, there is an open set U such that $\nabla f_1(q), ..., \nabla f_{n+1}(q)$ are linearly independent for any point $q \in U$. Since F is a nontrivial polynomial map there is some $q \in U$ such that $dF(q) \neq 0$. By the chain rule, $\nabla f_k \in \ker(dF)$ for all k. These vectors span $\boldsymbol{C}^{n+1}$, forcing $dF(q) = 0$, a contradiction. ♠

Let $\omega = \exp(2\pi i/n)$. For $v \in [1, (n-3)/2]$ let Λ_v be the collection of all sequences $\{s_1, ..., s_v\} \subset \{2, 3, ..., n-2\}$ such that $s_j \leq s_{j+1} + 2$ for all j. Define

$$\lambda_v = \sum_{I \in \Lambda_v} \omega^I; \qquad \omega^I = \prod_{j=1}^{v} \omega^{s_j}; \quad I = (s_i, ..., s_v). \tag{55}$$

In the next section we will prove that $\lambda_v \neq v$ for all $v \in [1, (n-3)/2]$. In this section we take this result, which we call the Vanishing Lemma, for granted.

We have two embeddings of $\boldsymbol{C}^n$ into $\boldsymbol{C}^{2n}$. Namely

$$o(z_1, ..., z_n) = (z_1, 0, z_2, 0, ..., z_n, 0); \quad e(z_1, ..., z_n) = (0, z_1, 0, z_2, ..., 0, z_n). \tag{56}$$

Define $H_k = O_k \circ o = E_k \circ e$. The function H_n is defined in $\boldsymbol{C}^n$.

Lemma 6.2 $\nabla H_1(p), ..., \nabla H_{(n-1)/2}(p), \nabla H_n(p)$ *are linearly independent at $p = (\omega, \omega^2, ..., \omega^n)$.*

Proof: Since H_{i+1} is cyclically invariant, and homogeneous of degree $i+1$, the gradient ∇H_i is homogeneous of degree i. Furthermore, the $(j+1)$th

entry of ∇H_i is obtained from the jth entry by shifting the indices of the variables by 1. These two facts imply that $\nabla H_{i+1}(p) = \mu_i V_i$, where

$$\mu_i = \frac{\partial H_i}{\partial z_n}\Big|_p; \qquad V_i = (\omega^i, \omega^{2i}, ..., \omega^{ni}).$$

The vectors $V_1, ..., V_{\frac{n-1}{2}}, V_n$ are certainly linearly independent over $\boldsymbol{C}$. It suffices to prove all the μ_i are nonzero.

As $H_1 = z_1 + ... + z_n$ and $H_n = z_1 ... z_n$, we have $\mu_1 = \mu_n = 1$. For the intermediate values of i, the terms in H_i have the form $(-1)^i z_{k_1}, ..., z_{k_i}$, where successive or repeating indices are not allowed. Note, in particular, that $...z_1 z_n...$ never occurs, because the notion of succession is reckoned cyclically. The terms in $\partial H_i / \partial z_n$ therefore have the form

$$(-1)^i z_{k_1}, ..., z_{k_{i-1}}; \qquad k_\alpha \leq k_{\alpha+1} + 2 \quad \forall \alpha.$$

Also, the terms z_{n-1}, z_n and z_1 do not occur. Thus, we see that $\mu_i = \pm \lambda_{i-1}$. Since $i \leq (n-1)/2$, the Vanishing Lemma says that $\mu_i \neq 0$. ♠

Lemma 6.3 *Let* $p_t = (t\omega, t\omega, t\omega^2, t\omega^2, ...t\omega^n, t\omega^n)$. *Then*

$$\lim_{t \to 0} t^{1-k} \nabla O_k(p_t) = o(\nabla H_k(p)); \qquad \lim_{t \to 0} t^{1-k} \nabla E_k(p_t) = e(\nabla H_k(p)).$$

Proof: We will derive the first equation, the second being similar. Given any point $q = (x_1, ..., x_{2n})$ define

$$S_t(q) = (t^{-1} x_1, t x_2, t^{-1} x_3, t x_4, t^{-1} x_5, ..., t x_{2n}). \tag{57}$$

We have

$$\lim_{t \to 0} S_t(p_t) = (\omega, 0, \omega^2, 0, ..., \omega^n, 0) = o(p). \tag{58}$$

Let $\partial_j O_k$ be the jth partial derivative of O_k. For any point q we have the general homogeneity:

$$O_k(S_t(q)) = t^{-k}(O_k(q)). \tag{59}$$

Hence

$$\partial_j O_k(S_t p_t) = t^{-k_j} \partial_j O_k(p_t) \qquad k_j = k + (-1)^j. \tag{60}$$

Combining Equations 58 and 60 we have

$$\lim_{t\to 0} t^{1-k}\partial_j O_k(p_t) = \begin{cases} \partial_j O_k(o(p)) & \text{j odd} \\ \\ 0 & \text{j even} \end{cases} \tag{61}$$

Finally,

$$\partial_{2j+1}(O_k(o(p)) = o(\partial_j H_k(p)). \tag{62}$$

Equations 61 and 62 together establish the first equation. ♠

We claim that the vectors $\nabla O_1(p_t), ..., \nabla E_n(p_t)$ are independent for some t. Otherwise there are functions $a_{k,t}$ and $b_{k,t}$ such that

$$\sum a_{k,t}\nabla O_k(p_t) + \sum b_{k,t}\nabla E_k(p_t) = 0; \qquad \max(|a_{k,t}|, ..., |b_{n,t}|) = 1. \tag{63}$$

We let $t \to 0$. Taking a subsequence, we can arrange that $\lim a_{k,t} = a_{k,0}$ and $\lim_{b,t} = b_{k,0}$ for all relevant k, with at least one limit being nonzero.

We multiply Equation 63 by t^{1-k} and take the limit using Lemma 6.3 to obtain

$$\sum a_{k,0}\ o(\nabla H_k(p)) + \sum b_{k,0}\ e(\nabla H_k(p)) = 0. \tag{64}$$

The subspaces $o(\boldsymbol{C}^n)$ and $e(\boldsymbol{C}^n)$ are orthogonal and the vectors $\{\nabla H_k(p)\}$ are linearly independent. This is a contradiction. Hence $\nabla O_1(p_t), ..., \nabla E_n(p_t)$ are algebraically independent for some value of t. Lemma 6.1 now completes our algebraic independence proof, modulo the Vanishing Lemma.

6.2 Proof of The Vanishing Lemma

We begin with some algebraic preliminaries. Say that an *adapted measure* is a positive measure τ, with integer sized atoms, supported in the nth roots of unity. (Here n is fixed, as above.) Each adapted measure τ is encoded by a non-decreasing sequence $I = \{s_1, ..., s_k\}$. The jth root of unity has τ-mass m iff j appears m times in I. We define $\langle\tau\rangle = \omega^I$, as in Equation 55. By convention, the $\langle\emptyset\rangle = 1$. We define the product $\tau_1 \cdot \tau_2$ to be the measure obtained by adding τ_1 and τ_2 together. (We will reserve the + symbol for another purpose.) Note that $\langle\tau_1 \cdot \tau_2\rangle = \langle\tau_1\rangle\langle\tau_2\rangle$. For $m \in \boldsymbol{N}$ we define $m\tau = \tau \cdot ... \cdot \tau$, a total of m times.

Let M denote the free abelian group generated by the adapted measures. A typical element of M is a finite formal sum $\sigma = \sum m_i \tau_i$. We define the *evaluation map*

$$\langle \sigma \rangle = \sum \langle \tau_i \rangle^{m_i}. \tag{65}$$

We make M into a ring using the product rule

$$(\sum_i m_i \sigma_i)(\sum_j n_j \sigma_j) = \sum_{i,j} m_i n_j (\sigma_i \cdot \tau_j) \tag{66}$$

The ring M is the *group ring* generated by the adapted measures. The evaluation map is a ring homomorphism from M to $\boldsymbol{C}$.

If $A \subset S^1$ is an arc and $v \in \boldsymbol{N}$ is an integer, let $\Psi(A, v) \subset M$ denote the sum, taken over all adapted measures which have mass v and are supported in A. Using the notation from the Vanishing Lemma, let $A_v \subset S^1$ be the open arc, containing -1, whose endpoints are ω^v and ω^{-v}.

Lemma 6.4 (Centrally Symmetric Compression) $\lambda_v = \langle \Psi(A_v, v) \rangle$.

Proof: Let Λ_v be the set of sequences used to define λ_v in Equation 55. If $I = (s_1, ..., s_v) \in \Lambda_v$, then let $\phi(I)$ be the summand of $\Psi(A_v, v)$ indexed by $(s_1 + v - 1, s_2 + v - 3, s_2 + v - 5, ..., s_{v-2} - v + 5, s_{v-1} - v + 3, s_v - v + 1)$. This map is a bijection which preserves the total sum of the elements in I, so that $\omega^I = \langle \phi(I) \rangle$. This lemma now follows from the definitions of the two relevant quantities. ♠

Our proof breaks down into two main cases, which we treat in turn.

6.2.1 Case 1: $v < n/4$

We say that a measure is *sparse* if it assigns at most mass 1 to any given point. For any pair (A, m) let $\Psi'(A, m)$ denote the formal sum of sparse mass-m measures, supported in A. Let $A^c = S^1 - A$.

Lemma 6.5 (Binomial Theorem) $\langle \Psi(A_v, v) \rangle = \langle \Psi'(A_v^c, v) \rangle$.

Proof: We write $A = A_v$, $A^c = A_v^c$, $\Psi = \Psi(A_v, v)$, and $\Psi' = \Psi'(A_v^c, v)$. Let Φ_j be the formal sum of all mass v measures whose support intersects A^c in exactly j points. Note that $\Phi_0 = \Psi$ and $\Phi_v = \Psi'$. Let Θ_j denote the formal

sum of all mass j adapted measures. By symmetry $\langle\Theta_j\rangle = 0$ for $j > 0$. Let $\Delta_j = \Psi_j(A^c, j)$ be the formal sum of sparse adapted measures of mass j which are supported in A^c. Note that $\Delta_v = \Psi'$.

Suppose that $k \in \{0, ..., v-1\}$. If $j \geq k$ and τ is a summand of Φ_j there are exactly j choose k ways to write $\tau = \tau_1 \cdot \tau_2$, where $\tau_1 \in \Delta_k$ and $\tau_2 \in \Theta_{v-k}$. The point is that we can choose the support of τ_1 to be any k-element subset of the A^c-support of τ. This way of counting things gives the relation:

$$\Delta_k \Theta_{v-k} = \sum_{j=k}^{v} \left(\begin{array}{c} j \\ k \end{array} \right) \Phi_j, \tag{67}$$

for $k = 0, ..., v-1$. Combining the previous equation with a familiar corollary of the binomial theorem,

$$\sum_{k=0}^{v-1} (-1)^k \Delta_k \Theta_{v-k} = \Phi_0 + (-1)^v \Phi_v. \tag{68}$$

Since $\langle\Delta_k \Theta_{v-k}\rangle = \langle\Delta_k\rangle\langle\Theta_{v-k}\rangle = 0$. we have $\langle\Psi\rangle = \langle\Phi_0\rangle = \pm\langle\Phi_v\rangle = \langle\Psi'\rangle$. ♠

Since $v < n/4$ we have $\Re(z) > 0$ for all $z \in A_v$. We will use induction to show that $\langle\Psi'(A_v, w)\rangle > 0$ for all $v, w \geq 1$. Let $\underline{\omega}^v$ be the mass 1 measure supported on ω^v. If τ is a mass w sparse measure supported in A_v^c then the support of τ intersects $\{\omega^v, \omega^{-v}\}$ in 0, 1, or 2 points. Thus

$$\Psi'(A_v^c, w) = \left\{ \begin{array}{c} \Psi'(A_{v-1}^c, w) \\ + \\ (\underline{\omega}^v + \underline{\omega}^{-v}) \cdot \Psi'(A_{v-1}^c, w-1) \\ + \\ (\underline{\omega}^v \cdot \underline{\omega}^{-v}) \cdot \Psi'(A_{v-1}^c, w-2). \end{array} \right\} \tag{69}$$

At least one term on the right is nontrivial. From

$$\langle\underline{\omega}^v + \underline{\omega}^{-v}\rangle = 2\Re(\omega^v) > 0; \qquad \langle\underline{\omega}^v \cdot \underline{\omega}^{-v}\rangle = 1. \tag{70}$$

and induction, any nontrivial term on the right hand side of Equation 69 evaluates to a positive number. Therefore, the left hand side evaluates to a positive number as well.

6.2.2 Case 2: $v \geq n/4$

For each integer $w \in (0, n/4]$ we choose an open arc B_w, invariant under complex conjugation, such that $-1 \in B_w$ and there are exactly w nth roots of unity contained in B_w. Let $\Psi(w, k', k)$ denote the formal sum of adapted mass k measures μ such that μ is supported in B_w and $\mu(B_w - B_{w-2}) \leq k'$.

Our goal is to show that $\langle \Psi(w, v, v) \rangle \neq 0$, where w is the number of nth roots of unity in A_v. We order the triples (w, k', k) lexicographically. We will show inductively that $\langle \Psi(w, k', k) \rangle > 0$ if k is even and $\langle \Psi(w, k', k) < 0$ if k is odd. (These sums are real, by symmetry.)

If $k = 1$ then $\langle \Psi(w, k', k) \rangle$ is the sum of numbers all of which have negative real part, so that $\langle \Psi(w, k', k) \rangle < 0$ in this case. Also, $\langle \Psi(1, k, k) \rangle = (-1)^k$. Henceforth we assume that $w \geq 2$ and $k \geq 2$. Since $w \geq 2$ there are two nth roots of unity α_1 and $\alpha_2 = \overline{\alpha}_1$ in $B_w - B_{w-2}$.

Suppose $w = 2$. A simple counting argument gives

$$\Psi(w, k, k) = (\underline{\alpha}_1 + \underline{\alpha}_2) \cdot \Psi(v, k-1, k-1) + \underline{\alpha}_1 \cdot \underline{\alpha}_2 \cdot \Psi(v, k-2, k-2).$$

Note that $\alpha_1 + \alpha_2 < 0$. By induction, both terms on the right have the desired sign when evaluated. Henceforth we assume that $w \geq 3$.

Suppose that $k' = 1$. A counting argument gives

$$\Psi(w, 1, k) = \Psi(w-2, k, k) + (\underline{\alpha}_1 + \underline{\alpha}_2) \cdot \Psi(w-2, k-1, k-1)$$

Again, we note that $\alpha_1 + \alpha_2 < 0$. Since $w \geq 3$ both terms on the right have the desired sign when evaluated.

Suppose that $k' = 2$. A counting argument gives

$$\Psi(w, 2, k) = \Psi(w-2, k, k) + (\underline{\alpha}_1 + \underline{\alpha}_2) \cdot \Psi(1, k-1) + \underline{\alpha}_1 \cdot \underline{\alpha}_2 \cdot \Psi(w-2, k-2, k-2).$$

By induction, all terms on the right have the desired sign when evaluated.

Suppose that $k' \geq 3$. A counting argument gives

$$\Psi(w, k', k) = \left\{ \begin{array}{c} \Psi(w-2, k'-2, k) \\ + \\ (\underline{\alpha}_1 + \underline{\alpha}_2) \cdot \Psi(w, k'-1, k-1) \\ + \\ \underline{\alpha}_1 \cdot \underline{\alpha}_2 \cdot \Psi(w, k'-2, k'-2) \end{array} \right\}.$$

By induction, all three terms on the right have the desired sign when evaluated.

This completes our proof.

7 References

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